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A Method in the Calculus of Variations.

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INTRODUCTION.

The Calculus of Variations originated in the discussion of the problem of finding the curve of quickest descent, the brachistochrone. This problem was proposed to other mathematicians by John Bernoulli* in 1696, and modifications of it have been discussed ever since. There was essentially but one method used by various mathematicians before Lagrange in dealing with this problem, a method developed most fully by Euler, who applied it to the general problem of minimizing a definite integral. The essentials of this method consist in

- 1) Considering a definite integral as the limit of a sum and a differential equation as the limit of an equation in differences;
- 2) Using the fact that the first derivative of a function vanishes for values of the variable minimizing the function.

Although Euler put his work in geometric form, his argument may be stated more tersely in analytic form as follows:

The problem is to find a function $y(x)$ such that $y = y(x)$ minimizes $\int_{x_0}^{x_n} F(x, y, dy/dx) dx$. Divide the interval $x_0 x_n$ into n equal parts of length Δ and use the notation $x_i = x_0 + i\Delta$. Then consider the problem of determining $(y_0, y_1, \dots, y_n)$ such as to minimize

$\Phi(y_0, y_1, \dots, y_n) \equiv F(x_1, y_1, p_1)\Delta + F(x_2, y_2, p_2)\Delta + \dots + F(x_n, y_n, p_n)\Delta,$
where $p_i = (y_i - y_{i-1})/\Delta$. A set of necessary conditions is that

$$\frac{\partial \Phi}{\partial y_i} \equiv \Delta F_y(x_i, y_i, p_i) - \{F_p(x_{i+1}, y_{i+1}, p_{i+1}) - F_p(x_i, y_i, p_i)\} = 0, \quad i=1, 2, \dots, n.$$

Then Euler reasoned that since an integral is the limit of a sum and a differential equation is the limit of a difference equation, the function $y(x)$ minimizing the integral must satisfy

$$F_y(x, y(x), y'(x)) - \frac{d}{dx} F_p(x, y(x), y'(x)) = 0.$$

* See Ostwald's *Klassiker der Exakten Wissenschaften*, Nr. 46.

Other methods have shown the conclusion to be correct. Furthermore, Knezer* has proved this by a modification of Euler's method.

When Lagrange† introduced the term "variation" and the suggestive notation " δ " to represent it and showed that $d(\delta f) = \delta(df)$, analytic methods replaced the former geometric methods, with a gain in rigor and possibility of further extension of the theory. However, it would seem that this turn of affairs probably prevented the early recognition of as close connections between the problem of the Calculus of Variations and the ordinary problem of maxima and minima as actually exist.

The first part of this paper is a development of the well known tests for a minimum of a function of several variables. It is introduced here because the methods of obtaining these tests are to be used later. The second part may be thought of as a comparison of an algebraic problem of minimizing a sum with the transcendental problem of minimizing a definite integral, the sum and the integral corresponding as in Euler's investigations. The theorems of Part II may be stated roughly as follows:

I. If, as Δ decreases indefinitely, the solution $(y_1, \dots, y_{n-1})$ of the algebraic problem approaches uniformly to corresponding ordinates of a curve $y=y(x)$, where $y(x)$ is continuous together with its first two derivatives in the interval under consideration, then this curve $y=y(x)$ satisfies the first three necessary conditions‡ for a solution of the transcendental problem.

II. If the absolute minimum of the algebraic problem approaches a curve with a continuously turning tangent, this curve gives the absolute minimum of the transcendental problem if such exists.

In Part III the converse problem is considered; it is shown that if $y(x)$ minimizes the transcendental problem, then for n sufficiently large a solution of the algebraic problem exists as close as we please to $y(x)$.

In Part IV the problem of the Calculus of Variations is considered following formal analogies with the algebraic problem. An important characteristic of the method used in the algebraic problem is that the variations $(\eta_1, \dots, \eta_{n-1})$ are unrestricted. To get a similar state of affairs in the transcendental problem of minimizing $\int_{x_0}^{x_n} F(x, y, y') dx$, we let the second derivative play the rôle of independent variable. This leads to the entrance of multiple integrals and shows clearly certain formal analogies between the algebraic and transcendental problems.

* "Euler und die Variationsrechnung," *Abhandlungen zur Geschichte der Mathematischen Wissenschaften*, XXV.

† Ostwald's *Klassiker*, Nr. 47.

‡ See Bolza, "Lectures on the Calculus of Variations," pp. 101, 102.

I. THE MINIMUM OF A FUNCTION OF SEVERAL VARIABLES.

Consider $\phi(x_1, x_2, \dots, x_n)$ a real function of the real variables $(x_1, \dots, x_n)$ of class* C''' in a certain neighborhood R of the origin, $|x_i| < k, i=1, 2, \dots, n$. Then for points $(x_1, \dots, x_n)$ within this region† R ,

$$\begin{aligned} \text{I. } \phi(x_1, \dots, x_n) - \phi(0, \dots, 0) &= \sum_{i=1}^n x_i \phi_i(0, \dots, 0) \\ &\quad + \frac{1}{2!} \sum_{i=1}^n \sum_{j=1}^n x_i x_j \phi_{ij}(0, \dots, 0) + l^2(l), \end{aligned}$$

in which we use the notation $l^2 = x_1^2 + \dots + x_n^2$, (l) represents a quantity infinitesimal with l , and

$$\phi_i(x_1, \dots, x_n) \equiv \frac{\partial \phi}{\partial x_i}(x_1, \dots, x_n); \quad \phi_{ij}(x_1, \dots, x_n) \equiv \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x_1, \dots, x_n).$$

To find necessary conditions that $\phi(x_1, \dots, x_n)$ have a relative minimum at $(0, \dots, 0)$, we assume that $(0, \dots, 0)$ gives a relative minimum. Then for $(x_1, \dots, x_n)$ sufficiently near the origin,

$$\phi(0, \dots, 0) \leq \phi(x_1, \dots, x_n)$$

and the left-hand side of I is positive or zero. In particular,

$$\begin{aligned} \text{I'. } \phi(x_1, 0, \dots, 0) - \phi(0, \dots, 0) &= x_1 \phi_1(0, \dots, 0) \\ &\quad + \frac{1}{2!} x_1^2 \phi_{11}(0, \dots, 0) + x_1^2(x_1) \geq 0. \end{aligned}$$

In order that this should not change sign with x_1 , we must have $\phi_1(0, \dots, 0) = 0$. In this way we obtain

$$1) \quad \phi_1(0, \dots, 0) = 0, \quad \phi_2(0, \dots, 0) = 0, \quad \dots, \quad \phi_n(0, \dots, 0) = 0.$$

Then I becomes

$$\text{II. } \phi(x_1, \dots, x_n) - \phi(0, \dots, 0) = \frac{1}{2!} \sum_{i=1}^n \sum_{j=1}^n x_i x_j \phi_{ij}(0, \dots, 0) + l^2(l),$$

and I' becomes

$$\text{II'. } \phi(x_1, 0, \dots, 0) - \phi(0, \dots, 0) = \frac{1}{2!} x_1^2 \phi_{11}(0, \dots, 0) + x_1^2(x_1).$$

For x_1 sufficiently small the sign on the right in II' depends on the sign of $\phi_{11}(0, \dots, 0)$, if $\phi_{11}(0, \dots, 0) \neq 0$. Thus as a second set of necessary con-

* $\phi(x_1, \dots, x_n)$ is said to be of class $C^{(m)}$ in a given region if ϕ , together with all its partial derivatives of orders 1, 2, $\dots$, m , is continuous in this region.

† See Goursat, "Cours d'analyse, t. I, p. 119.

II. THE ALGEBRAIC PROBLEM AND ITS LIMIT.

Consider a real function of three independent real variables, $F(x, y, p)$ of class C''' in a certain simply connected region R of the xy plane and for all values of p . Let (x_0, y_0) , (x_n, y_n) be two interior points of R , $x_0 < x_n$; let the equation $x_0 + n\Delta = x_n$, n being a positive integer, determine the quantity Δ , and let $x_i = x_0 + i\Delta$, $i=1, 2, \dots, n-1$. Now consider the sum

$$\begin{aligned}\phi(y_1, \dots, y_{n-1}) &= F\left(x_1, y_1, \frac{y_1 - y_0}{\Delta}\right)\Delta + F\left(x_2, y_2, \frac{y_2 - y_1}{\Delta}\right)\Delta \\ &\quad + \dots + F\left(x_n, y_n, \frac{y_n - y_{n-1}}{\Delta}\right)\Delta \\ &= \sum_{i=1}^n F\left(x_i, y_i, \frac{y_i - y_{i-1}}{\Delta}\right)\Delta,\end{aligned}$$

$y_1, \dots, y_{n-1}$ being such that the points (x_i, y_i) lie entirely in R . This is obviously a function of precisely the type considered in Part I. We propose to consider for this particular function the conditions for a minimum derived in Part I, and at the same time to recapitulate the essentials of the argument.

For n a fixed positive integer suppose that $(Y_1, \dots, Y_{n-1})$, such that (x_i, Y_i) is an interior point of R , $i=1, 2, \dots, n-1$, gives a relative minimum for $\phi(y_1, \dots, y_{n-1})$. Then if $\eta_1, \dots, \eta_{n-1}$ are any $n-1$ real quantities, a positive number δ can be found such that when $|\varepsilon| < \delta$

$$1) \quad \phi(Y_1 + \varepsilon\eta_1, \dots, Y_{n-1} + \varepsilon\eta_{n-1}) - \phi(Y_1, \dots, Y_{n-1}) \geq 0.$$

Expanding by Taylor's series with a remainder (this expansion is legitimate with our hypotheses on F) and using the abbreviated bracket notation,

$$\begin{aligned}& \left[x_i, Y_i, \frac{Y_i - Y_{i-1}}{\Delta} \right] = (i), \\ 2) \quad & \sum_{i=1}^n \left\{ F\left(x_i, Y_i + \varepsilon\eta_i, \frac{Y_i - Y_{i-1} + \varepsilon(\eta_i - \eta_{i-1})}{\Delta}\right) - F\left(x_i, Y_i, \frac{Y_i - Y_{i-1}}{\Delta}\right) \right\} \Delta \\ &= \varepsilon \sum_{i=1}^n \left\{ \eta_i F_y(i) + \frac{\eta_i - \eta_{i-1}}{\Delta} F_p(i) \right\} \Delta \\ &+ \frac{\varepsilon^2}{2!} \sum_{i=1}^n \left\{ \eta_i^2 F_{yy}(i) + 2\eta_i \frac{\eta_i - \eta_{i-1}}{\Delta} F_{yp}(i) + \left(\frac{\eta_i - \eta_{i-1}}{\Delta} \right)^2 F_{pp}(i) \right\} \Delta + \varepsilon^2(\varepsilon) \geq 0,\end{aligned}$$

in which $\eta_0 = \eta_n = 0$ and (ε) indicates a quantity which is infinitesimal with ε .

Since the right-hand side of equation 2) must not change sign with ε , for any real values $(\eta_1, \dots, \eta_{n-1})$ we must have

$$3) \quad \sum_{i=1}^n \left\{ \eta_i F_y(i) + \frac{\eta_i - \eta_{i-1}}{\Delta} F_p(i) \right\} \Delta = 0.$$

Then the coefficient of each η in 3) must vanish. We can get the coefficient of η_i in 3) very readily by inspection, but to make the analogy to the method

of the Calculus of Variations more evident we use the formula for partial summation:

$$\sum_{i=p}^w b_i(a_i - a_{i-1}) \equiv \sum_{i=p}^w a_i(b_i - b_{i+1}) - a_{p-1}b_p + a_w b_{w+1}.$$

Since $\eta_0 = \eta_n = 0$, this identity gives us

$$\sum_{i=1}^n \left(\frac{\eta_i - \eta_{i-1}}{\Delta} \right) F_p(i) = \sum_{i=1}^n \eta_i \left(\frac{F_p(i) - F_p(i+1)}{\Delta} \right),$$

and 3) becomes

$$\sum_{i=1}^n \eta_i \left[F_y(i) - \frac{F_p(i+1) - F_p(i)}{\Delta} \right] \Delta = 0.$$

Hence in this case conditions 1), Part I, are

$$F_y(i) - \frac{F_p(i+1) - F_p(i)}{\Delta} = 0, \quad i=1, 2, \dots, n-1.$$

Also in 2) the coefficient of ε^2 must be positive or zero; i. e.,

$$4) \quad \sum_{i=1}^n \left\{ \eta_i^2 P_i + 2\eta_i \frac{\eta_i - \eta_{i-1}}{\Delta} Q_i + \left(\frac{\eta_i - \eta_{i-1}}{\Delta} \right)^2 R_i \right\} \Delta \geq 0,$$

in which $P_i \equiv F_{yy}(i)$; $Q_i \equiv F_{yp}(i)$; $R_i \equiv F_{pp}(i)$.

Let j be a positive integer less than n , and choose $\eta_i = 0$, $i \neq j$, $\eta_j \neq 0$. (This is analogous to a method often useful in the Calculus of Variations of giving the minimizing curve an increment for a very short distance). Then the coefficient of η_j^2 in 4) must be greater than or equal to zero, giving condition 2), Part I, for this particular problem,

$$\Delta P_j + 2Q_j + \frac{R_{j+1} + R_j}{\Delta} \geq 0, \quad j=1, 2, \dots, n-1.$$

Treating the quadratic form 4) in a manner closely analogous to that used in dealing with the second variation in the general algebraic problem, Part I, we have immediately,

$$A. \quad \frac{\sum_{i=1}^n \left\{ \eta_i^2 P_i + 2\eta_i \frac{\eta_i - \eta_{i-1}}{\Delta} Q_i + \left(\frac{\eta_i - \eta_{i-1}}{\Delta} \right)^2 R_i \right\}}{\sum_{i=1}^n \eta_i^2 \Delta} \geq 0.$$

Since the expression A is homogeneous of degree zero in the η 's, it has the same value for the set $(\eta_1, \dots, \eta_{n-1})$ as for $(\rho\eta_1, \dots, \rho\eta_{n-1})$. Then it takes all its values on the locus $\eta_1^2 + \dots + \eta_{n-1}^2 = 1/\Delta$, a closed locus, and therefore it assumes its lower and its upper limits, and its lower limit is positive or zero. At points where extremal values are taken we have, from A,

$$\sum_{i=1}^n \eta_i^2 \Delta \frac{\partial}{\partial \eta_i} (\text{numerator of } A) - \text{numerator of } A \cdot 2\eta_i \Delta = 0, \quad i=1, 2, \dots, n-1.$$

For such points on $\sum_{i=1}^n \eta_i^2 \Delta = 1$ (or $\sum_{i=1}^{n-1} \eta_i^2 \Delta = 1$, since $\eta_n = 0$),

$$\frac{\partial}{\partial \eta_i} (\text{numerator of } A) = 2\eta_i \Delta \cdot \lambda,$$

where λ is the extremal value of the fraction at the point in question. Performing the indicated differentiation and using the notation $u_i - u_{i-1} = \delta u_i$,

$$\eta_i P_i - \eta_i \frac{\delta Q_{i+1}}{\Delta} - \frac{R_{i+1} \frac{\delta \eta_{i+1}}{\Delta} - R_i \frac{\delta \eta_i}{\Delta}}{\Delta} + Q_i \frac{\delta \eta_i}{\Delta} - Q_{i+1} \frac{\delta \eta_{i+1}}{\Delta} = \eta_i \lambda, \quad i=1, 2, \dots, n-1;$$

or, re-arranging,

$$\text{B. } R_{i+1} \frac{\frac{\delta \eta_{i+1}}{\Delta} - \frac{\delta \eta_i}{\Delta}}{\Delta} + \frac{\delta R_{i+1}}{\Delta} \frac{\delta \eta_i}{\Delta} + \left(\frac{\delta Q_{i+1}}{\Delta} - P_i + \lambda \right) \eta_i + Q_{i+1} \frac{\delta \eta_{i+1}}{\Delta} - Q_i \frac{\delta \eta_i}{\Delta} = 0, \\ i=1, 2, \dots, n-1.$$

If $(Y_1, \dots, Y_{n-1})$ gives a relative minimum, system B can have no solution $(\eta_1, \dots, \eta_{n-1}) \not\equiv 0$ for $\lambda < 0$. It should be remembered that $\eta_0 = \eta_n = 0$.

Summarizing: If $(Y_1, Y_2, \dots, Y_{n-1})$ gives a minimum for $\phi(y_1, \dots, y_{n-1})$, then

$$\text{I'. } F_y(i) - \frac{F_p(i+1) - F_p(i)}{\Delta} = 0, \quad i=1, 2, \dots, n-1.$$

$$\text{II'. } \Delta^2 P_i + 2\Delta \cdot Q_i + R_{i+1} + R_i \geq 0, \quad i=1, 2, \dots, n-1.$$

III'. The set of equations B have no solution $(\eta_1, \dots, \eta_{n-1}) \not\equiv 0$ for $\lambda < 0$ such that $\eta_0 = \eta_n = 0$.

Also, as in the general case, Part I, we have as sufficient conditions for a relative minimum, conditions I', II' and

III''. The set of equations B have no solution $(\eta_1, \dots, \eta_{n-1}) \not\equiv 0$, such that $\eta_0 = \eta_n = 0$, for $\lambda \leq 0$.

We wish now to see what becomes of these conditions I', II', III' as n , the number of divisions of our interval, increases indefinitely, assuming that the set $(Y_1, \dots, Y_{n-1})$ approach the corresponding ordinates of a curve of class C'' , $y(x)$; we hope in this way to show that $y(x)$ satisfies certain necessary conditions for minimizing the definite integral, $\int_{x_0}^{x_n} F(x, y, dy/dx) dx$.

Let x' be a fixed abscissa of our interval, $x_0 < x' \leq x_n$, and for any particular division of our interval let j be such that $x_{j-1} < x' \leq x_j$. It is evident that j increases indefinitely with n , while $\lim_{j \rightarrow \infty} x_j = x'$; this will be indicated thus:

$$\lim_{n \rightarrow \infty, j \rightarrow \infty} x_j = x'.$$

Assume that, as we increase the number of points of division x_i , the ordinates $(Y_1, \dots, Y_{n-1})$ approach uniformly to the corresponding values of a function $y(x)$ of class C'' in the interval $x_0 \leq x \leq x_n$; i. e., given $\varepsilon > 0$, N can be found independent of i such that whenever the number of divisions n of the interval (x_0, x_n) is greater than N ,

$$|Y_i - y(x_i)| < \varepsilon \text{ for } i=1, 2, \dots, n-1.$$

Then because of the uniform continuity of $y(x)$ in the closed interval (x_0, x_n) , given $\varepsilon > 0$, N can be found independent of x' such that whenever $n > N$, $|Y_j - y(x')| < \varepsilon$ for any x' of our interval.

Assume furthermore that $\text{Lt}_{n=\infty} (Y_i - Y_{i-1})/\Delta = y'(x_i)$, and that this limit is approached uniformly. Then because of the uniform continuity of $y'(x)$ in (x_0, x_n) , $\text{Lt}_{n=\infty, j=\infty} (Y_j - Y_{j-1})/\Delta = y'(x')$, and this limit is approached uniformly for all values of x' .

These assumptions, together with the assumption that F_y is a continuous function of its three arguments for (x, y) in region R and for all values of p , gives us that

$$\text{Lt}_{n=\infty, j=\infty} F_y \left[x_j, Y_j, \frac{Y_j - Y_{j-1}}{\Delta} \right] = F_y[x', y(x'), y'(x')],$$

and that this limit is approached uniformly for all values of x' involved. (When convenient, we shall denote the arguments $[x, y(x), y'(x)]$ by $[x]$.) Just as with F_y , it is evident that $F_p, F_{yy}, F_{yp}, F_{pp}, F_{ypx}, F_{ypp}, F_{ppx}, F_{ppy}, F_{ppp}$, with the arguments (j) , approach uniformly to the corresponding functions with the arguments $[x']$, in the interval $x_0 < x' \leq x_n$.

We shall now consider equations I' as n increases indefinitely:

$$\text{I'.} \quad \Delta F_y(i) + F_p(i) - F_p(i+1) = 0, \quad i=1, 2, \dots, n-1.$$

We wish to show that $y(x)$ satisfies Euler's differential equation; i. e., that

$$\text{I.} \quad F_y[x] - \frac{d}{dx} F_p[x] = 0.$$

From I' we have

$$5) \quad F_p(1) - F_p(j) + \sum_{i=1}^{j-1} F_y(i) \cdot \Delta = 0.$$

Since $\text{Lt}_{n=\infty, j=\infty} x_j = x'$, and since $F_y(i)$ approaches $F_y[x_1]$ uniformly as n increases indefinitely,

$$\text{Lt}_{n=\infty, j=\infty} \sum_{i=1}^{j-1} F_y(i) \cdot \Delta = \int_{x_0}^{x'} F_y[x] dx.$$

Also

$$\text{Lt}_{n=\infty} F_p(1) = F_p[x_0]; \quad \text{Lt}_{n=\infty, j=\infty} F_p(j) = F_p[x'].$$

Since the limit of a sum is the sum of the limits, it follows from 5) that

$$F_p[x_0] - F_p[x'] + \int_{x_0}^{x'} F_y[x] dx = 0.$$

Differentiating with respect to x' , we obtain equation I for x in the interval $x_0 < x < x_n$. Since F_y and $\frac{d}{dx} F_p$ are continuous, it follows that equation I holds for all values of x in the closed interval $x_0 \leq x \leq x_n$.

We may write I' in the form

$$F_y(j) - F_{px}(j + \theta_1) - \frac{Y_{j+1} - Y_j}{\Delta} F_{py}(j + \theta_2) - \frac{\frac{Y_{j+1} - Y_j}{\Delta} - \frac{Y_j - Y_{j-1}}{\Delta}}{\Delta} F_{pp}(j + \theta_3) = 0,$$

in which $(j + \theta_i)$ indicates that the arguments are taken between those indicated by (j) and by $(j + 1)$. We know that $\text{Lt}_{n=\infty, j=\infty}$ of each function involved in this last equation exists and is approached uniformly throughout the interval with the exception of the fraction in the last term. Then if we assume that $F_{pp}[x] \neq 0$ in the interval $x_0 \leq x \leq x_n$, i. e., that $F_{pp}(x, y, p) \neq 0$ along $y(x)$, we have

$$\text{Lt}_{n=\infty, j=\infty} \frac{\delta Y_{j+1} - \delta Y_j}{\Delta^2} = \frac{F_y[x'] - F_{px}[x'] - y'(x') F_{py}[x']}{F_{pp}[x']}.$$

From equation I we obtain

$$y''(x') = \frac{F_y[x'] - F_{px}[x'] - y'(x') F_{py}[x']}{F_{pp}[x']}.$$

Hence,

$$6) \quad \text{Lt}_{n=\infty, j=\infty} \frac{\delta Y_{j+1} - \delta Y_j}{\Delta^2} = y''(x'),$$

and furthermore this limit is approached uniformly for $x_0 < x' < x_n$.

Consider condition II', which we rewrite,

$$\text{II}'. \quad \Delta^2 P_j + 2\Delta Q_j + R_{j+1} + R_j \geq 0.$$

It is evident that

$$\text{Lt}_{n=\infty, j=\infty} \{ \Delta^2 P_j + 2\Delta Q_j + R_{j+1} + R_j \} = 2F_{pp}[x'].$$

Thus we get, as the limiting form of the second condition,

$$\text{II}. \quad F_{pp}[x] \equiv R[x] \geq 0, \quad x_0 \leq x \leq x_n.$$

Assuming as above that $F_{pp}[x] \neq 0$, we have

$$\text{II}'''. \quad F_{pp}[x] > 0.$$

We shall now use condition III' to show that if $u(x)$ is a solution of

$$\psi(u) \equiv \frac{d}{dx} (Ru') - (P - Q')u = 0,$$

such that $u(x_0)=0$, $u(x)\not\equiv 0$ (the arguments of P, Q', R being $[x]$), then $u(x)$ has no root between $x=x_0$ and $x=x_n$; i. e., $u(x)\not\equiv 0$, in the interval $x_0 < x < x_n$.

We shall make use here of the two following theorems: *

a) Given the system

$$\begin{cases} \frac{d}{dx}\left(k(x)\frac{du}{dx}\right) + [\lambda g(x) - l(x)]u = 0, \\ u(a) = \alpha', \\ u'(a) = \alpha, \end{cases}$$

in which $k(x) > 0$, $g(x) > 0$, $k(x)$ is of class C' , $a \leq x \leq b$ and $g(x)$, $l(x)$ are continuous in the same interval, and α, α' are constants not both zero; there is one and only one value of the parameter λ for which the solution of this system has a specified number (including zero) of roots between a and b and is such that $u(b) = 0$.

b) Given the systems

$$\begin{cases} \frac{d}{dx}\left(\kappa_1 \frac{du}{dx}\right) - G_1 u = 0, \\ u(a) = 0, \end{cases} \quad \begin{cases} \frac{d}{dx}\left(\kappa_2 \frac{du}{dx}\right) - G_2 u = 0, \\ u(a) = 0, \end{cases}$$

in which κ_1, κ_2 are functions of x of class C' , and G_1, G_2 are continuous in the interval $a \leq x \leq b$; furthermore, $G_2 < G_1$ and $0 < \kappa_2 \leq \kappa_1$. Let $u_1(x)$ be a non-identically vanishing solution of the first system and $u_2(x)$ likewise for the second. If $u_1(x)$ has k roots in the interval $a < x \leq b$, then $u_2(x)$ has at least k roots in this interval and the i -th root of $u_2(x)$ is smaller than the i -th root of $u_1(x)$.

Suppose that $\psi(u) = 0$ has a non-identically vanishing solution, $u(x)$, such that $u(x_0) = 0$, $u(x') = 0$ for some number x' such that $x_0 < x' < x_n$. To prove our theorem we shall show that this supposition leads to a contradiction; in fact it leads to a solution of the difference equation system B, $(\eta_1, \dots, \eta_{n-1}) \not\equiv 0$, such that $\eta_0 = \eta_n = 0$, for λ negative.

Making use of the above theorems a), b), our present supposition gives us that there is one and only one value $\lambda_0 < 0$ such that the solution of the system

$$7) \quad \begin{cases} \psi(u) + \lambda_0 u = 0, \\ u(x_0) = 0, \\ u'(x_0) = \gamma, \end{cases}$$

* Given in this form by Professor Bôcher in lectures. See also in article by Bôcher, *Bulletin of Amer. Math. Soc.*, Vol. IV, p. 298, I, and p. 304, VII, and *Transactions of Amer. Math. Soc.*, Vol. I, p. 418.

in which γ is a positive constant, has $x=x_n$ as its first root to the right of $x=x_0$. If we replace λ_0 by $\lambda_0-\delta$, $\delta>0$, the solution of system 7), $u(x, \lambda_0-\delta)$ does not vanish in the interval $x_0 \leq x \leq x_n$ and $u(x_n)$ is positive; for λ_0 replaced by $\lambda_0+\delta$ the solution $u(x, \lambda_0+\delta)$ vanishes within the interval (x_0, x_n) , and for δ sufficiently small, $u(x_n)$ is negative.

The plan is first to show that we can get one solution of the difference system B, with $\eta_0=0$, as near to $u(x, \lambda_0-\delta)$ as we please and therefore with $\eta_n>0$, and one as near to $u(x, \lambda_0+\delta)$ as we please and therefore with $\eta_n<0$. Then by holding n fast and varying λ between $\lambda_0+\delta$ and $\lambda_0-\delta$ we shall get a solution of B such that $\eta_0=\eta_n=0$, for some negative λ , which contradicts condition III'.

The solution of the linear differential system

$$8) \quad \begin{cases} \psi(u) + \lambda u = 0, \\ u(x_0) = 0, \\ u'(x_0) = \gamma \end{cases}$$

can be obtained by the Cauchy-Lipschitz method as the limit approached by the solution of the linear difference equation system (using our previous notation, $\delta\eta_i = \eta_i - \eta_{i-1}$),

$$9) \quad \begin{cases} R[x_i] \frac{\delta^2 \eta_{i+2}}{\Delta^2} + R'[x_i] \frac{\delta \eta_{i+1}}{\Delta} + [Q'[x_i] - P[x_i]] \eta_i + \lambda \eta_i = 0, \\ \eta_0 = 0, \\ \eta_1 = \Delta \gamma. \end{cases} \quad i=0, 1, 2, \dots, n-2,$$

The theorem involved here may be stated as follows:

Given $f(x, y, v)$, $g(x, y, v)$ real continuous functions of the three real variables (x, y, v) in the region R : $|x-x_0|<a$, $|y-y_0|<b$, $|v-v_0|<b$, satisfying the inequalities

$$\begin{aligned} |f(x, y', v') - f(x, y, v)| &< k_1 |y' - y| + k_2 |v' - v|, \\ |g(x, y', v') - g(x, y, v)| &< k_1 |y' - y| + k_2 |v' - v|, \end{aligned}$$

k_1, k_2 being constants and (x, y', v') , (x, y, v) being points in R ; then there exist solutions $y(x)$, $v(x)$ of the differential equations

$$\frac{dy}{dx} = f(x, y, v), \quad \frac{dv}{dx} = g(x, y, v),$$

satisfying the initial conditions $y(x_0)=y_0$, $v(x_0)=v_0$. These functions are approached by the solutions of the difference equation system

$$\begin{cases} y_{i+1} - y_i = (x_{i+1} - x_i) f(x_i, y_i, v_i), \\ v_{i+1} - v_i = (x_{i+1} - x_i) g(x_i, y_i, v_i), \end{cases} \quad i=1, 2, \dots, n.$$

in which $x_{n+1} \equiv x$ is a point of the interval $|x - x_0| < k$ in which the solutions of the differential equations are continuous, $x_1, x_2, \dots, x_n$ being points of the interval (x_0, x) ; the approach mentioned occurs when the number of divisions of the interval (x_0, x) is increased indefinitely and the length of the longest interval (x_i, x_{i+1}) is decreased indefinitely.*

Furthermore, the solutions of the difference system approach $y(x), v(x)$ uniformly in any interval which, with its end points, lies within the largest interval in which the solutions $y(x), v(x)$ of the differential equations are continuous and f, g satisfy the conditions originally imposed.†

An examination of the proof of this theorem shows that it holds with no essential changes if in the difference equations we replace $f(x_i, y_i, v_i), g(x_i, y_i, v_i)$ by functions which approach these values uniformly for all values of x' in the interval of convergence of the solution $y(x), v(x)$ when i increases with n in such a way that $\lim_{n \rightarrow \infty} x_i = x'$. It is of course essential that the subscripts in the difference equation system be such that y_i, v_i can be determined in succession when y_0, v_0 are given.

It is this generalization of the theorem which is of use in the present problem.

The differential equation $\psi(u) + \lambda u = 0$ is equivalent to the system

$$10) \quad \begin{cases} \frac{du}{dx} = v & \equiv f(x, y, v), \\ \frac{dv}{dx} = \frac{(P - Q' - \lambda)u - R'v}{R} & \equiv g(x, y, v), \end{cases}$$

and the difference equation system 9) is equivalent to the system

$$11) \quad \begin{cases} \eta_{i+1} - \eta_i = (x_{i+1} - x_i)v_i, \\ v_{i+1} - v_i = (x_{i+1} - x_i) \frac{[P[x_i] - Q'[x_i] - \lambda]\eta_i - R'[x_i]v_i}{R[x_i]}, \end{cases}$$

which evidently corresponds to the above system of differential equations in applying the Cauchy-Lipschitz theorem. The functions f, g satisfy a Lipschitz condition since $R \neq 0$ and f, g have continuous first partial derivatives with regard to x, u, v in the neighborhood of the solutions $u(x), v(x)$ satisfying the initial conditions $u(x_0) = 0, v(x_0) = \gamma$.

But instead of the difference system 9) we should like to be able to substitute system B, Part II. More precisely, we wish to show that for η sufficiently large, solutions of B exist uniformly close to corresponding solutions of system 8) for certain values of λ .

* Picard, "Traité d'analyse," t. 2, p. 322.

† Picard, "Traité d'analyse," t. 2, p. 332, and Lipschitz, "Lehrbuch der Analysis," Bd. 2, p. 504.

System B is equivalent to

$$12) \quad \begin{cases} \eta_{i+1} - \eta_i = (x_{i+1} - x_i) v_i, \\ v_{i+1} - v_i = \\ (x_{i+1} - x_i) \frac{\left(P_{i+1} - \frac{Q_{i+2} - Q_{i+1}}{\Delta} - \lambda \right) (\eta_i + \Delta v_i) - \left(\frac{R_{i+2} - R_{i+1}}{\Delta} + Q_{i+2} - Q_{i+1} \right) v_i}{R_{i+2} + \Delta Q_{i+2}}, \end{cases} \quad i=0, 1, 2, \dots, n-2.$$

From the extension mentioned of the Cauchy-Lipschitz theorem we see that we can replace system 11) by system 12) if we can show that, as n increases, P_{i+1} , $(Q_{i+2} - Q_{i+1})/\Delta$, $(R_{i+2} - R_{i+1})/\Delta$ approach uniformly to $P[x_i]$, $Q'[x_i]$, $R'[x_i]$.

That P_{i+1} approaches uniformly to $P[x_{i+1}]$ has already been shown. Then P_{i+1} approaches uniformly to $P[x_i]$ because of the uniform continuity of $P[x]$ in the interval $x_0 \leq x \leq x_n$.

To show that $(Q_{i+2} - Q_{i+1})/\Delta$ approaches uniformly to $Q'[x_i]$, we note that

$$\begin{aligned} Q_{i+2} - Q_{i+1} &= (x_{i+2} - x_{i+1}) Q_x(i+1+\theta_1) + (Y_{i+2} - Y_{i+1}) Q_y(i+1+\theta_2) \\ &\quad + \left(\frac{Y_{i+2} - Y_{i+1}}{\Delta} - \frac{Y_{i+1} - Y_i}{\Delta} \right) Q_p(i+1+\theta_3), \end{aligned}$$

in which $(i+1+\theta)$ means that the arguments are between $(i+2)$ and $(i+1)$. Dividing through by $x_{i+2} - x_{i+1}$, i. e. by Δ , and using the information regarding limits including equation 6), we see that $\lim_{n=\infty, i+2=\infty}$ of the right-hand side exists.

$$\therefore \lim_{n=\infty, i+2=\infty} \frac{Q_{i+2} - Q_{i+1}}{\Delta} = Q_x[x_{i+1}] + y'(x_{i+1}) Q_y[x_{i+1}] + y''(x_{i+1}) Q_p[x_{i+1}];$$

i. e.,

$$\lim_{n=\infty, i+2=\infty} (Q_{i+2} - Q_{i+1})/\Delta = Q'[x_{i+1}].$$

Furthermore, the limits on the right are approached uniformly, and from this fact, together with the uniform continuity of $y(x)$, $y'(x)$ and $Q'(x)$ in the interval $x_0 \leq x \leq x_n$, it follows that $(Q_{i+2} - Q_{i+1})/\Delta$ approaches uniformly to $Q'[x_i]$, $i=1, 2, \dots, n-2$, when n increases indefinitely. Exactly similar reasoning gives similar results for $(R_{i+2} - R_{i+1})/\Delta$.

Thus the solution of system 8) is approached by the solution of the difference equation system

$$13) \quad \begin{cases} R_{i+1} \frac{\frac{\delta r_{i+1}}{\Delta} - \frac{\delta \eta_i}{\Delta}}{\Delta} + \frac{\delta R_{i+1}}{\Delta} \cdot \frac{\delta \eta_i}{\Delta} + \left(\frac{\delta Q_{i+1}}{\Delta} - P_i + \lambda \right) \eta_i \\ \quad + Q_{i+1} \frac{\delta \eta_{i+1}}{\Delta} - Q_i \frac{\delta \eta_i}{\Delta} = 0, \quad i=1, 2, \dots, n-1. \\ \eta_0 = 0, \quad \eta_1 = \gamma \cdot \Delta. \end{cases}$$

Furthermore, since the coefficients P, Q', R' in the differential system 10) are of class C' and $F_{pp} \neq 0$ in a region of the xy plane including the locus $y=y(x)$, $x_0 \leq x \leq x_n$ and for all values of the third argument, a solution of system 11) or 12) approaches a corresponding solution of system 10) uniformly in the interval $x_0 \leq x \leq x_n$, when n increases indefinitely; and similarly the solutions of system 13) approach uniformly to the solutions of system 8) as n increases indefinitely.

Then for $\lambda = \lambda_0 - \delta$ and n sufficiently large the solutions of 13) are always positive, $i > 0$ and $\eta_n(\lambda_0 - \delta) > 0$, while for $\lambda = \lambda_0 + \delta$ and n sufficiently large, $\eta_n(\lambda_0 + \delta) < 0$. Now choose $n = N$ sufficiently large that $\eta_n(\lambda_0 - \delta) > 0$, $\eta_n(\lambda_0 + \delta) < 0$ and hold n fast. Then $\eta_n(\lambda)$ is a continuous function of λ , and for some λ' in the interval $\lambda_0 - \delta < \lambda < \lambda_0 + \delta$, and therefore negative, we have $\eta_n(\lambda') = 0$. Thus, if there is a solution of $\psi(u) = 0$, vanishing at $x = x_0$ and again at $x = x'$, where $x_0 < x' < x_n$, and not vanishing identically, then there exists a λ' less than zero such that in the corresponding solution $(\eta_0 \eta_1 \dots \eta_{n-1} \eta_n)$, not identically zero, the variable η_n is zero. But this contradicts condition III', and therefore a solution $u(x)$ of $\psi(u) = 0$ such that $u(x_0) = 0$ and $u(x) \neq 0$ can not vanish in the interval $x_0 < x < x_n$.

The results of this work can be summarized in the following

THEOREM I. *If the following conditions hold,*

- 1) $(Y_1 Y_2 \dots Y_{n-1})$ such that $(x_1 Y_1), \dots, (x_{n-1} Y_{n-1})$ lie entirely in the interior of region R give a relative minimum for $\Phi(y_1 y_2 \dots y_{n-1})$ in R ;
- 2) As n increases indefinitely $(Y_1 Y_2 \dots Y_{n-1})$ tend uniformly toward the corresponding ordinates of a curve $y(x)$ of class C'' lying entirely in the interior of R and going through the fixed points $(x_0, y_0), (x_n, y_n)$;
- 3) When j is allowed to increase with n in such a way that x_j approaches a definite value x' , $\lim_{n \rightarrow \infty, j \rightarrow \infty} (Y_j - Y_{j-1}) / \Delta$ exists and is equal to

$$dy(x)/dx|_{x=x'};$$

- 4) $F_{pp}[x, y(x), y'(x)] \neq 0$, $x_0 \leq x \leq x_n$ ($y'(x) = dy(x)/dx$):

then it follows that

I. $y(x)$ satisfies Euler's differential equation,

$$F_y[x, y(x), y'(x)] - \frac{d}{dx} F_p[x, y(x), y'(x)] = 0.$$

II. $R > 0$ along* $y(x)$; i. e., $F_{pp}[x, y(x), y'(x)] > 0$, $x_0 \leq x \leq x_n$.

* Notice that under hypotheses 1, 2, 3 we have proved that $F_{pp}[x, y(x), y'(x)] \geq 0$.

III. If $u(x)$ is a solution of Jacobi's equation,

$$\frac{d}{dx} (Ru') - (P - Q')u = 0,$$

vanishing for $x = x_0$ but not vanishing identically, then $u(x)$ does not vanish in the interval $x_0 < x < x_n$.

Thus we see that $y(x)$ satisfies the first three necessary conditions* usually given for a function giving a relative minimum of $\int_{x_0}^{x_n} F(x, y, y') dx$.

It is well to notice that we can make this theorem somewhat stronger by replacing the first hypothesis by the part of it which we have really used; namely, instead of assuming that $(Y_1 \dots Y_{n-1})$ furnishes a relative minimum for ϕ in R , we need merely assume the three necessary conditions for a relative minimum in the algebraic problem.

We note also that our conclusions are not sufficient conditions for $y(x)$ to give even a weak minimum of $\int_{x_0}^{x_n} F(x, y, y') dx$. Furthermore, if we assume sufficient conditions for a proper minimum in the algebraic problem, and nothing more, we should not be surprised at being unable to show that the conjugate point† x' is beyond x_n , for the form A may be positive for any particular n and still approach zero as n increases indefinitely. However, if we assume sufficient conditions for a proper minimum in the algebraic problem and assume furthermore that as n increases indefinitely the least value λ of A approaches a positive limit, not zero, then our method shows that the conjugate point x' is beyond x_n if existent and our conclusions are sufficient conditions that $y(x)$ give a weak minimum for $\int_{x_0}^{x_n} F(x, y, y') dx$.

The above discussion leads to the consideration of the following

THEOREM II. If we assume,

- 1) $(Y_1 \dots Y_{n-1})$ give the absolute minimum for $\sum_{i=1}^n F[x_i, y_i, (y_i - y_{i-1})/\Delta]$, (x_i, y_i) being in region R ;
- 2) As n increases indefinitely, $(Y_1 \dots Y_{n-1})$ approach uniformly the values of the corresponding ordinates of a curve $y = y(x)$ lying in the interior of R , of class C' ;
- 3) As j increases with n in such a way that $\lim_{n=\infty} x_j = x'$,

$$\lim_{n=\infty, j=\infty} (Y_j - Y_{j-1})/\Delta$$

exists and is equal to $y'(x')$, and this limit is approached uniformly in the interval $x_0 < x' \leq x_n$:

* See Bolza, "Lectures," pp. 101, 102.

† For definition of conjugate point, see Bolza, "Lectures," p. 60.

then it follows that

- I. $\lim_{n \rightarrow \infty} \sum_{i=1}^n F[x_i, Y_i, (Y_i - Y_{i-1})/\Delta] \Delta = \int_{x_0}^{x_n} F[x, y(x), y'(x)] dx.$
- II. If there is a curve of class C' lying entirely within R and giving the absolute minimum for $\int_{x_0}^{x_n} F(x, y, y') dx$, then $y(x)$ is such a curve.

PROOF: From hypotheses 2), 3) and the uniform continuity of $F(x, y, p)$ for (x, y) in R and p varying between any two fixed limits, it follows that as n increases indefinitely $F[x_i, Y_i, (Y_i - Y_{i-1})/\Delta]$ approaches uniformly to $F[x_i, y(x_i), y'(x_i)]$, $i=1, 2, \dots, n$. From this in turn follows conclusion I.

Let $\bar{y}_1 \dots \bar{y}_{n-1}$ be the ordinates of any other curve $y = \bar{y}(x)$ corresponding to the abscissas $x_1, \dots, x_{n-1}$, $\bar{y}(x)$ being of class C' and $y = \bar{y}(x)$ being interior to region R . We wish to show that

$$\int_{x_0}^{x_n} F(x, \bar{y}(x), \bar{y}'(x)) dx \geq \int_{x_0}^{x_n} F(x, y, y') dx.$$

For n sufficiently large $\sum_{i=1}^n F[x_i, \bar{y}_i, (\bar{y}_i - \bar{y}_{i-1})/\Delta] \Delta$ is as near as we please to $\int_{x_0}^{x_n} F(x, \bar{y}, \bar{y}') dx$. But from the first hypothesis

$$\sum_{i=1}^n F\left(x_i, \bar{y}_i, \frac{\bar{y}_i - \bar{y}_{i-1}}{\Delta}\right) \Delta \geq \sum_{i=1}^n F\left(x_i, Y_i, \frac{Y_i - Y_{i-1}}{\Delta}\right) \Delta.$$

Then using conclusion I, we read from this inequality conclusion II.

III. FROM THE TRANSCENDENTAL PROBLEM TO THE ALGEBRAIC PROBLEM.

Although sufficient conditions for a solution of the algebraic problem do not give a solution of the limiting transcendental problem, it is interesting to note that if, conversely, $y(x)$ satisfies sufficient conditions for a weak minimum of $\int_{x_0}^{x_n} F(x, y, y') dx$, then for n sufficiently large we can find a solution of the algebraic problem. To prove this we consider first the

THEOREM: If the following conditions hold,

- 1) $y = y(x)$ is a solution of Euler's equation, of class C'' in the interval $x_0 \leq x \leq x_n$, going through the points (x_0, y_0) , (x_n, y_n) and lying entirely in the interior of the region* R ;
- 2) The conjugate point to x_0 is beyond x_n or non-existent;
- 3) F_{pp} is positive along $y(x)$;
- 4) $y(x)$ is of class C' in an interval $x_0 - \delta \leq x \leq x_n + \delta$, where δ is some positive constant:

then, given $\epsilon > 0$, N exists, independent of i , such that whenever $n > N$ a solution of the system

* The notation here is the same as at the beginning of Part II.

$$\text{I.} \quad \begin{cases} F_y(i) - \frac{F_p(i+1) - F_p(i)}{\Delta} = 0, & i=1, 2, \dots, n-1, \\ y_0 = y(x_0) \end{cases}$$

exists, satisfying the relations

$$\text{a)} \quad \begin{cases} |y_i - y(x_i)| < \varepsilon, \\ \left| \frac{y_i - y_{i-1}}{\Delta} - y'(x_i) \right| < \varepsilon, & i=1, 2, \dots, n-1, \end{cases}$$

$$\text{b)} \quad y_n = y(x_n).$$

We shall show first that for n sufficiently large the system

$$\text{I'.} \quad \begin{cases} F_y(i) - \frac{F_p(i+1) - F_p(i)}{\Delta} = 0, & i=1, 2, \dots, n-1, \\ y_0 = y(x_0), \\ \frac{y_1 - y_0}{\Delta} = y'(x_0) \end{cases}$$

has a unique solution satisfying conditions a).

Let x, y, p be taken as the rectangular coördinates of a point in space. Since F_{pp} is positive along $y(x)$, we can construct a "tube" S including as interior points all the points $[x, y(x), y'(x)]$ and such that $F_{pp}(x, y, p)$ is positive for all points of this tube.* Let

$$1) \quad v = F_p(x, y + p \cdot \Delta, p)$$

define a transformation of the points of the four-space $(x, y, p; \Delta)$ such that (x, y, p) is a point in S and Δ is a small positive quantity, into points of $(x, y, v; \Delta)$ space, and let L denote the image of $y(x)$, i. e., of $[x, y(x), y'(x); 0]$. We shall denote the locus $[x, y(x), y'(x); 0]$ by k . The function $F_p(x, y + p \cdot \Delta, p)$ is of class C' in a region of $(x, y, p; \Delta)$ space including as interior points the points of the locus k . Differentiating this function with respect to p , we have

$$F_{pp}(x, y + p \cdot \Delta, p) + \Delta \cdot F_{py}(x, y + p \cdot \Delta, p),$$

which is positive along the closed interior locus k . All the hypotheses of the extended theorem of inversion† are fulfilled, so that

a) A positive quantity ρ can be found so small that to each point $(x, y, v; \Delta)$ in (L, ρ) , the image of the ρ -neighborhood of k, k_ρ , there corresponds one and but one point $(x, y, p; \Delta)$ of k_ρ .

b) The inverse function ϕ ,

$$2) \quad p = \phi(x, y, v; \Delta),$$

is of class C' in (L, ρ) .

* See Bolza, "Vorlesungen über Variationsrechnung," p. 157, Lemma II.

† See Bolza, "Vorlesungen," p. 160, b; p. 163, c.

c) Corresponding to ρ we can find a positive quantity σ such that to each point of the σ -neighborhood of L , L_σ there corresponds, through the transformation 2), one and only one point in k_ρ .

If there is a solution of I' satisfying conditions a) as desired, choose Δ so small that $[x_i, y, (y_i - y_{i-1})/\Delta; \Delta]$ lies in k_ρ , $i = 1, 2, \dots, n-1$. Let $F_p(x_i, y_{i-1} + \Delta p_i, p_i) = v_{i-1}$, where $p_i \equiv (y_i - y_{i-1})/\Delta$. Then from 2) we have $p_i = \Phi(x_i, y_{i-1}, v_{i-1}; \Delta)$, and the solution of system I' lying in k_ρ becomes a solution of the following system lying in (L, ρ) :

$$\text{II.} \quad \begin{cases} v_i - v_{i-1} = (x_{i+1} - x_i) F_y \{ x_i, y_{i-1} + \Delta \cdot \Phi(x_i, y_{i-1}, v_{i-1}; \Delta), \\ \quad \Phi(x_i, y_{i-1}, v_{i-1}; \Delta) \} \equiv \Delta \cdot f_1(x_i, y_{i-1}, v_{i-1}; \Delta), \\ y_i - y_{i-1} = (x_{i+1} - x_i) \Phi(x_i, y_{i-1}, v_{i-1}; \Delta), \\ y_0 = y(x_0); (y_1 - y_0)/\Delta = y'(x_0). \end{cases}$$

Conversely, if II has a solution in L_σ , then I' has a solution in k_ρ . Thus the system II is equivalent to system I'.

Euler's differential equation,

$$F_y(x, y, y') - \frac{d}{dx} F_p(x, y, y') = 0,$$

is equivalent to the system

$$\text{III.} \quad \begin{cases} \frac{dv}{dx} = F_y[x, y, \Phi(x, y, v)] \equiv g(x, y, v), \\ \frac{dy}{dx} = \Phi(x, y, v) \equiv f(x, y, v), \end{cases}$$

in which $\Phi(x, y, v) \equiv \Phi(x, y, v; 0)$. Then our hypothesis is that $[v(x), y(x)]$ is a set of solutions of III, $y(x)$ being the original extremal and $v(x) = F_p[x, y(x), y'(x)]$. The functions $g(x, y, v)$, $f(x, y, v)$ satisfy a Lipschitz condition in the sub-region of (L, ρ) in which $\Delta = 0$, since they are of class C' in this region. Then the solution $[v(x), y(x)]$ of III is approached by a solution of the difference equation system,

$$\text{IV.} \quad \begin{cases} v_i - v_{i-1} = (x_i - x_{i-1}) F_y[x_{i-1}, y_{i-1}, \Phi(x_{i-1}, y_{i-1}, v_{i-1})], \\ y_i - y_{i-1} = (x_i - x_{i-1}) \Phi(x_{i-1}, y_{i-1}, v_{i-1}), \quad i = 1, 2, \dots, n-1. \end{cases}$$

Comparing this system with II, we note that since $\Phi(x, y, v; \Delta)$ is of class C' in (L, ρ) and F_y is continuous in its three arguments in the region of interest, the functions on the right in II approach uniformly to the corresponding functions on the right in IV as n increases indefinitely, $i = 1, 2, \dots, n-1$. Then the generalization of the Cauchy-Lipschitz theorem, specified in Part II, informs us that for n sufficiently large system II has a solution as near as we

please to $[v(x), y(x)]$; i. e., given $\varepsilon > 0$, N can be found such that whenever $n > N$,

$$|y_i - y(x_i)| < \varepsilon, \quad |v_i - v(x_i)| < \varepsilon.$$

Furthermore, N can be chosen independent of i . Here we use the fact that the Cauchy-Lipschitz method gives, for n sufficiently large, solutions of a difference equation system uniformly close to the solution of the differential equation system in any interval which, together with its end points, lies in an interval in which the solutions of the differential system are continuous. We have assumed $y(x)$ to be of class C' for $x_0 - \delta \leq x \leq x_n + \delta$, $\delta > 0$. That $v(x)$ is continuous in such an interval follows from this restriction on $y(x)$ and the continuity of F_p for (x, y) in R and for all values of p .

We have, also,

$$\frac{y_i - y_{i-1}}{\Delta} - y'(x_i) = \phi(x_i, y_{i-1}, v_{i-1}; \Delta) - \phi(x_i, y(x_i), v(x_i); 0).$$

Since ϕ is continuous in (L, ρ) , N exists independent of i such that

$$\left| \frac{y_i - y_{i-1}}{\Delta} - y'(x_i) \right| < \varepsilon, \quad i = 1, 2, \dots, n,$$

whenever $n > N$. It is evident that the solutions of systems II and I' are unique.

Having established the existence of a unique solution of I' satisfying conditions a), we are in a position to use the idea of a set of extremals through a point to prove the theorem stated above. Since we assume that the conjugate point of x_0 is beyond x_n or non-existent, $y(x)$ is one of a set of extremals* $y(x, \gamma)$ through (x_0, y_0) , γ being the slope of the curve $y = y(x, \gamma)$ for $x = x_0$, which do not intersect in the interval $x_0 < x < x_n$. Let us write $\gamma_0 = y'(x_0)$. Then there exists a positive number d_1 such that when $|\gamma - \gamma_0| \geq d_1$ the corresponding extremals do not intersect in the interval $x_0 < x \leq x_n$. Also a positive number d_2 exists such that when $|\gamma - \gamma_0| \geq d_2$, F_{pp} is positive along $y(x, \gamma)$. Then the hypotheses for the application of the Cauchy-Lipschitz theorem are satisfied in the neighborhood of any one of the curves $y(x, \gamma')$, where $|\gamma_0 - \gamma'| < d$, d being the smaller of d_1 and d_2 . Furthermore, an examination of the Cauchy-Lipschitz proof shows that the theorem applies when the initial value of v , v_0 is made a function of the parameter γ , $|\gamma_0 - \gamma| < d$, and therefore we have that for a given $\varepsilon > 0$, N_1 can be found independent of i and γ such that, whenever $n > N_1$,

$$\begin{aligned} |y_i(\gamma) - y(x_i, \gamma)| &< \varepsilon, \\ \left| \frac{y_i(\gamma) - y_{i-1}(\gamma)}{\Delta} - y'(x_i, \gamma) \right| &< \varepsilon, \quad i = 1, 2, \dots, n; \quad |\gamma - \gamma_0| < d, \end{aligned}$$

in which the set $y_1(\gamma), \dots, y_n(\gamma)$ is the solution of I' for $y'(x_0) = \gamma$.

* See Bolza, "Lectures," p. 61.

Our conclusion is that for n sufficiently large we are able to get the solution of the difference system I' uniformly close to the corresponding extremal throughout the interval $x_0 \leq x \leq x_n$ and for $|\gamma - \gamma_0| < d$, γ being the initial slope of the extremal in question. We are now in a position to prove the original theorem; i. e., to find a solution of the system I satisfying conditions a) and b).

As a function of γ alone, $y(x, \gamma)$ and $y'(x, \gamma)$ are continuous for $|\gamma - \gamma_0| < d$. Then a positive number d_3 exists such that

$$\begin{aligned} |y(x, \gamma) - y(x, \gamma_0)| &< \varepsilon/2, \\ |y'(x, \gamma) - y'(x, \gamma_0)| &< \varepsilon/2, \end{aligned}$$

whenever $|\gamma - \gamma_0| \leq d_3$. Let γ_1, γ_2 be such that $\gamma_1 < \gamma_0 < \gamma_2$, $\gamma_2 - \gamma_0 < d_3$, $\gamma_0 - \gamma_1 < d_3$. As a function of γ alone, $y(x, \gamma)$ is an increasing function, since the set of extremals through (x_0, y_0) do not intersect in the interval in question, and there exists a positive number $h < \varepsilon/2$ such that

$$\begin{aligned} y(x_n, \gamma_2) - y(x_n, \gamma_0) &> h, \\ y(x_n, \gamma_0) - y(x_n, \gamma_1) &> h. \end{aligned}$$

Now choose N_2 so large that for $n > N_2$

$$\begin{aligned} |y_i(\gamma) - y(x_i, \gamma)| &< h, \\ \left| \frac{y_i(\gamma) - y_{i-1}(\gamma)}{\Delta} - y'(x_i, \gamma) \right| &< h, \quad \gamma_1 \leq \gamma \leq \gamma_2. \end{aligned}$$

Choose $n = n' > N_2$ and hold it fast. Then the inequalities $y_{n'}(\gamma_1) < y_{n'}(\gamma_0) < y_{n'}(\gamma_2)$ hold. Also $y_{n'}(\gamma)$ is a continuous function of γ , $|\gamma - \gamma_0| < d_2$. Then for some value of γ , $\gamma = \gamma'$ such that $\gamma_1 < \gamma' < \gamma_2$, $y_{n'}(\gamma') = y(x_n, \gamma_0)$. Furthermore, $|y_i(\gamma') - y(x_i, \gamma')| < \varepsilon/2$ and $|y(x_i, \gamma') - y(x_i, \gamma_0)| < \varepsilon/2$.

$$\therefore |y_i(\gamma') - y(x_i, \gamma_0)| < \varepsilon,$$

and similarly,

$$\left| \frac{y_i(\gamma') - y_{i-1}(\gamma')}{\Delta} - y'(x_i, \gamma_0) \right| < \varepsilon.$$

Thus, in $y_1(\gamma'), \dots, y_{n'}(\gamma')$ we have a solution of system I satisfying conditions a) and b).

To carry out the plan of Part III we wish now to show that for n sufficiently large a minimum of the algebraic problem exists. We shall show that $(y_1, \dots, y_{n-1})$ exist satisfying the sufficient conditions I', II' III" of Part II for the desired minimum. We already have the existence for n sufficiently large of a solution $(y_1, \dots, y_{n-1})$ of

$$\text{I.} \quad F_y(i) - \frac{F_p(i+1) - F_p(i)}{\Delta} = 0, \quad i=1, 2, \dots, n-1,$$

going through (x_0, y_0) , (x_n, y_n) and as near to the extremal $y(x)$ as we please. To show that, for n sufficiently large, condition

$$\text{II.} \quad \Delta^2 P_i + 2\Delta Q_i + R_{i+1} + R_i > 0, \quad i=1, 2, \dots, n-1,$$

is satisfied, we note that we can make R_i, R_{i+1} as near to $F''_{pp}[x_i, y(x_i), y'(x_i)]$ as we please and therefore positive, and that P_i, Q_i remain finite. It is sufficient now to show that the difference equation system B, Part II, has no solution $(\eta_0, \eta_1, \dots, \eta_n)$ not vanishing identically, such that $\eta_0 = \eta_n = 0$ for $\lambda \leq 0$. (It should be noticed that the equality sign in $\lambda \leq 0$ is especially important.)

We have in Part II that for n sufficiently large a solution of the difference system B such that $\eta_0 = 0, (\eta_1, \dots, \eta_{n-1}) \not\equiv 0$ is uniformly close to a corresponding solution of the system

$$3) \quad \begin{cases} \frac{d}{dx}(R\eta') + (Q' - P + \lambda)\eta = 0, \\ \eta(x_0) = 0; \quad \eta(x) \not\equiv 0. \end{cases}$$

But by hypothesis the conjugate point to x_0 is beyond x_n or non-existent. This means that if solutions of 3) for $\lambda = 0$ vanish for any x greater than x_0 , the next root, x' , is greater than x_n ; then a theorem* of Sturm's tells us that for $\lambda < 0$ the next root would be still further to the right. It is therefore evident that for n sufficiently large no solution of the difference system exists for $\lambda \leq 0$ such that $\eta_0 = \eta_n = 0$ and $(\eta_1, \dots, \eta_{n-1}) \not\equiv 0$. Thus for n sufficiently large we have the existence of $(y_1, \dots, y_{n-1})$ satisfying the sufficient conditions stated in Part II for a relative minimum of the algebraic sum. Furthermore, it is well to note that for n sufficiently large the ordinates $(y_1, \dots, y_{n-1})$ are as close as we please to the extremal giving the assumed minimum of the transcendental problem.

IV. THE TRANSCENDENTAL PROBLEM FOLLOWING FORMAL ANALOGIES TO THE ALGEBRAIC PROBLEM.

There is evidently a very close correspondence between the algebraic and the transcendental problems which we have considered. The purpose of this part of the paper is to make this correspondence still more striking by bringing out the formal analogy involved.

Let $F(x, y, p)$ be a function of class C''' in a certain simply connected region R of the xy plane including as interior points the points $(0, 0)$ and $(1, 0)$, and for all values of p . Let $y(x)$ be a function of class C'' in the interval $0 \leq x \leq 1$, such that the curve $y = y(x)$ lies entirely in the interior of R and goes through the points $(0, 0)$ and $(1, 0)$. Let

* See Theorem b), Part II.

$$L(y) \equiv p_0(x) \frac{d^2 y}{dx^2} + p_1(x) \frac{dy}{dx} + p_2(x) \cdot y$$

be a self-adjoint differential expression such that the system

$$\begin{cases} L(y) = 0, \\ y(0) = y(1) = 0 \end{cases}$$

is incompatible, and let $G(x, \xi)$ be the corresponding Green's function.* Suppose we note that $L[y(x)] = r(x)$. Let $\eta(x)$ be continuous in the interval $0 \leq x \leq 1$, and let $\bar{y}(x)$ be the function determined by

$$\begin{aligned} L(\bar{y}) &= r(x) + \epsilon \eta(x), \\ \bar{y}(0) &= \bar{y}(1) = 0, \end{aligned}$$

where ϵ is a constant chosen small enough that $\bar{y}(x)$ lies in the interior of R . Making use of the Green's function, we may write

$$y(x) = \int_0^1 r(\xi) G(x, \xi) d\xi; \quad y'(x) = \int_0^1 r(\xi) G_1(x, \xi) d\xi,$$

where $G_1(x, \xi) = \frac{\partial}{\partial x} G(x, \xi)$;

$$\bar{y}(x) = \int_0^1 [r(\xi) + \epsilon \eta(\xi)] G(x, \xi) d\xi; \quad \bar{y}'(x) = \int_0^1 [r(\xi) + \epsilon \eta(\xi)] G_1(x, \xi) d\xi;$$

$$a) \quad \bar{y}(x) - y(x) = \epsilon \int_0^1 \eta(\xi) G(x, \xi) d\xi; \quad \bar{y}'(x) - y'(x) = \epsilon \int_0^1 \eta(\xi) G_1(x, \xi) d\xi.$$

Using Taylor's series with a remainder,

$$\begin{aligned} F(x, \bar{y}, \bar{y}') &= F(x, y, y') + (\bar{y} - y) F_y + (\bar{y}' - y') F_{y'} \\ &\quad + \frac{1}{2!} \{ (\bar{y} - y)^2 F_{yy} + 2(\bar{y} - y)(\bar{y}' - y') F_{yp} + (\bar{y}' - y')^2 F_{pp} \} \\ &\quad + \frac{1}{3!} \{ (\bar{y} - y)^3 F_{yyy} + \dots + (\bar{y}' - y')^3 F_{ppp} \}, \end{aligned}$$

the arguments of all partial derivatives except in the last group being (x, y, y') , those of the last group being of the form $[x, y + \theta_1(\bar{y} - y), y' + \theta_2(\bar{y}' - y')]$, θ_1, θ_2 being proper fractions. Substituting the values of $\bar{y} - y, \bar{y}' - y'$ from a), we have

$$\begin{aligned} F(x, \bar{y}, \bar{y}') &= F(x, y, y') + \epsilon \{ F_y \int_0^1 \eta(\xi) G(x, \xi) d\xi + F_{y'} \int_0^1 \eta(\xi) G_1(x, \xi) d\xi \} \\ &\quad + \frac{\epsilon^2}{2!} \{ F_{yy} (\int_0^1 \eta(\xi) G(x, \xi) d\xi)^2 + 2F_{yp} \int_0^1 \eta(\xi) G(x, \xi) d\xi \int_0^1 \eta(\xi) G_1(x, \xi) d\xi \\ &\quad \quad \quad + F_{pp} (\int_0^1 \eta(\xi) G_1(x, \xi) d\xi)^2 \} + \epsilon^2(\epsilon) \\ &= F(x, y, y') + \epsilon \int_0^1 \eta(\xi) [G(x, \xi) F_y + G_1(x, \xi) F_{y'}] d\xi \\ &\quad + \frac{\epsilon^2}{2!} \int_0^1 \int_0^1 \eta(\xi_1) \eta(\xi_2) [G(x, \xi_1) G(x, \xi_2) F_{yy} + 2G(x, \xi_1) G_1(x, \xi_2) F_{yp} \\ &\quad \quad \quad + G_1(x, \xi_1) G_1(x, \xi_2) F_{pp}] d\xi_1 d\xi_2 + \epsilon^2(\epsilon). \end{aligned}$$

* For the definition of the *Green's function* see an article by Bôcher, *Annals of Mathematics*, Vol. XIII, p. 71.

Suppose $y(x)$ makes $\int_0^1 F(x, y, y') dx$ as small as or smaller than does any other curve of class C'' in a certain neighborhood of $y = y(x)$ and going through the points $(0, 0)$ and $(1, 0)$. Then for ε sufficiently small

$$\int_0^1 [F(x, \bar{y}, \bar{y}') - F(x, y, y')] dx \geq 0.$$

Employing reasoning which was useful in Parts I, II, we obtain conditions exactly analogous to conditions there derived,

$$\text{I"}. \quad \int_0^1 \left\{ \int_0^1 \eta(\xi) [G(x, \xi) F_y + G_1(x, \xi) F_p] d\xi \right\} dx = 0,$$

$$\text{II"}. \quad \int_0^1 \left\{ \int_0^1 \int_0^1 \eta(\xi_1) \eta(\xi_2) [G(x, \xi_1) G(x, \xi_2) F_{yy} + 2G(x, \xi_1) G_1(x, \xi_2) F_{yp} + G_1(x, \xi_1) G_1(x, \xi_2) F_{pp}] d\xi_1 d\xi_2 \right\} dx \geq 0.$$

Up to this point we have made no essential use of the fact that the third argument in F is the derivatives of $y(x)$, or in any way related to $y(x)$. We should have come to conditions analogous to I" and II" if the third argument in F had been any other function of x and had been varied similarly to the way in which $y(x)$ was varied. So far the important fact has been the possibility of expansion by means of Taylor's series with a remainder, *i. e.*, an expansion in ascending powers of a quantity ε which can be made as small as we please. From the very definition of maximum and minimum it is evident that such an expansion is fundamental in a general analytic investigation of such problems, whether the variable be a point or a curve.

Since $G(x, \xi)$ is continuous in the square $0 \leq x \leq 1$, $0 \leq \xi \leq 1$, and $G_1(x, \xi)$ is continuous in this square except along the diagonal $x = \xi$, where it has a unit discontinuity, the order of integration may be reversed in I" and II", giving

$$\text{I'}. \quad \int_0^1 \eta(\xi) \left[\int_0^1 \{ G(x, \xi) F_y + G_1(x, \xi) F_p \} dx \right] d\xi = 0,$$

$$\begin{aligned} \text{II'}. \quad & \int_0^1 \int_0^1 \eta(\xi_1) \eta(\xi_2) \left[\int_0^1 \{ G(x, \xi_1) G(x, \xi_2) P \right. \\ & \left. + 2G(x, \xi_1) G_1(x, \xi_2) Q + G_1(x, \xi_1) G_1(x, \xi_2) R \} dx \right] d\xi_1 d\xi_2 \\ & \equiv \int_0^1 \int_0^1 \eta(\xi_1) \eta(\xi_2) \Phi(\xi_1, \xi_2) d\xi_1 d\xi_2 \geq 0, \end{aligned}$$

where $\Phi(\xi_1, \xi_2)$ represents the integral in brackets [].

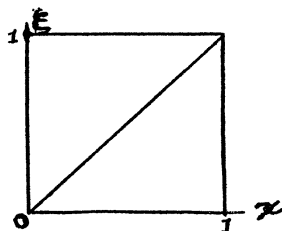
Since I' is an expression for the first variation of $\int_0^1 F(x, y, y') dx$, we should expect to be able to show from it that $y(x)$ is a solution of Euler's equation. To do this we proceed as follows:

$$\int_0^1 G_1(x, \xi) F_p dx = \int_0^\xi G_1(x, \xi) F_p dx + \int_\xi^1 G_1(x, \xi) F_p dx.$$

Since in each sub-interval $G_1(x, \xi)$ is continuous, we may integrate by parts thus:

$$\int_0^\xi G_1(x, \xi) F_p dx = G(x, \xi) F_p \Big|_0^\xi - \int_0^\xi G(x, \xi) \frac{d}{dx} (F_p) dx,$$

$$\int_\xi^1 G_1(x, \xi) F_p dx = G(x, \xi) F_p \Big|_\xi^1 - \int_\xi^1 G(x, \xi) \frac{d}{dx} (F_p) dx.$$



Since $G(x, \xi)$ is continuous and $G(0, \xi) = G(1, \xi) = 0$,

$$\int_0^1 G_1(x, \xi) F_p dx = - \int_0^1 G(x, \xi) \frac{d}{dx} (F_p) dx.$$

Then I' may be written,

$$1) \quad \int_0^1 \eta(\xi) \left[\int_0^1 G(x, \xi) \left\{ F_y - \frac{d}{dx} F_p \right\} dx \right] d\xi = 0.$$

Since 1) holds for $\eta(\xi)$ any continuous function, it holds if $\eta(\xi)$ is of class C' in the interval $0 \leq \xi \leq 1$ and $\eta(0) = \eta(1) = 0$. The coefficient of $\eta(\xi)$ in the integrand involved in 1), $\int_0^1 G(x, \xi) \left\{ F_y - \frac{d}{dx} F_p \right\} dx$, is a continuous function of ξ . Then, by the fundamental lemma* of the Calculus of Variations,

$$\int_0^1 G(x, \xi) \left\{ F_y - \frac{d}{dx} F_p \right\} dx = 0.$$

From this it follows that

$$I. \quad F_y[x, y(x), y'(x)] - \frac{d}{dx} F_p[x, y(x), y'(x)] = 0.$$

For, suppose $F_y - \frac{d}{dx} F_p = \phi(x) \neq 0$. Since $G(x, \xi)$ is symmetric, we have $\int_0^1 G(x, \xi) \phi(x) dx = 0$; but this would be the solution of the system

$$\begin{cases} L(u) = \phi(\xi), \\ u(0) = u(1) = 0, \end{cases}$$

which can not vanish identically since $\phi(x) \neq 0$. The contradiction proves the truth of I.

What we have really proved is that of all curves of class C'' through the points $(0, 0)$, $(1, 0)$, those giving a minimum must satisfy Euler's equation. We can not expect more of this method since we consider no more general curves. However, we shall assume in the discussion of the second variation that $F_{pp} \neq 0$ along $y(x)$; and it has been shown† that in this case an extremal of class C' is of class C'' .

It is interesting to notice that the first variation, I', is a simple integral with respect to ξ ; that the second variation, $\int_0^1 \int_0^1 \eta(\xi_1) \eta(\xi_2) \phi(\xi_1, \xi_2) d\xi_1 d\xi_2$, can be considered as a double integral; and it is evident that the k -th variation would be a k -fold integral. Here we have a complete formal analogy to the simple and multiple sums in the corresponding variations in the algebraic problem.

In examining the second variation, II', we shall prove first that $\phi(\xi, \xi) \geq 0$, $0 \leq \xi \leq 1$. This condition is analogous to the condition that $\phi_{ii} \geq 0$ in the algebraic problem,‡ and the method of proof is very similar.

* See Bolza, "Lectures," p. 20.

† Bolza, "Lectures," pp. 25, 26.

‡ See Part I, condition 2).

Suppose that for $\xi = x'$, $\phi(\xi, \xi) = -c$, where c is positive, x' being a point in the interval $0 < x < 1$. The function $\phi(\xi_1, \xi_2)$ is continuous in the square $0 \leq \xi_1 \leq 1$, $0 \leq \xi_2 \leq 1$, as can be seen by breaking up the integral representing $\phi(\xi_1, \xi_2)$ into integrals over intervals in which the integrand is continuous. Then we can find a positive quantity h such that $\phi(\xi_1, \xi_2) < 0$ in the square, $x' - h \leq \xi_1 \leq x' + h$, $x' - h \leq \xi_2 \leq x' + h$. Now choose $\eta(x)$ positive in the interval $x' - h < x < x' + h$, zero outside of this interval, and continuous in the interval $0 \leq x \leq 1$. Then we have the inequality

$$\int_0^1 \int_0^1 \eta(\xi_1) \eta(\xi_2) \phi(\xi_1, \xi_2) d\xi_1 d\xi_2 < 0,$$

which is contrary to condition II'.

$$\therefore \phi(\xi, \xi) \geq 0, \quad 0 \leq \xi \leq 1.$$

From this condition we shall show that if $R(x) \neq 0$ in the interval $0 \leq x \leq 1$, then $R(x)$ is positive in this interval. For this purpose we need the following properties of $G(x, \xi)$, $G_1(x, \xi)$:

1) $G(x, \xi)$ as a function of x satisfies the boundary conditions $u(0) = u(1) = 0$; i. e., $G(0, \xi) = G(1, \xi) = 0$.

2) $G(x, \xi)$ is symmetric. $\therefore G(x, 0) = G(x, 1) = 0$.

3) $G_1(x, \xi)_{x=\xi_+} - G_1(x, \xi)_{x=\xi_-} = 1$. ($x = \xi_+$ is read, "as x approaches ξ from the right.")

These facts enable us to show by direct computation that $\phi(0, 0) = 0$ and $\phi'(0, 0) = R(0)$, where $\phi'(\xi, \xi) \equiv d\phi(\xi, \xi)/d\xi$. We see that $\phi'(0, 0)$ is not negative, since if it were, $\phi(\xi, \xi)$ would be negative for very small values of ξ . $\therefore R(0) > 0$, since we assume $R(\xi) \neq 0$ in the interval $0 \leq \xi \leq 1$; then the continuity of $R(\xi)$ gives us the desired result,

$$\text{II.} \quad R(x) > 0,$$

in the interval $0 \leq x \leq 1$.

In the further consideration of the second variation we shall show that as the kernel* of an integral, $\phi(\xi_1, \xi_2)$ has only positive characteristic numbers. Suppose $\lambda = \lambda_0$ is a characteristic number and that

$$\eta(x) = \lambda_0 \int_0^1 \phi(x, \xi) \eta(\xi) d\xi.$$

Multiplying both sides by $\eta(x)$ and integrating,

$$\int_0^1 \eta^2(x) dx = \lambda_0 \int_0^1 \int_0^1 \phi(x, \xi) \eta(\xi) \eta(x) d\xi dx.$$

Since the left-hand side is positive and the double integral is not negative, and therefore actually positive in this case, λ_0 must be positive.

* The number λ is a characteristic number of a kernel $F(x, y)$ if a continuous function $u(x)$ exists such that $u(x) = \lambda \int_0^1 F(x, \xi) u(\xi) d\xi$. See also any book on integral equations.

To get Jacobi's condition we proceed as follows: Let $\bar{y}(x) - y(x) = \epsilon u(x)$, where $u(x)$ is of class C'' and $u(0) = u(1) = 0$. It is well known* that for such a variation on $y(x)$ the second variation of the integral can be written

$$2) \quad J(u) \equiv \int_0^1 u \psi(u) dx,$$

where† $\psi(u) \equiv (P - Q')u - d(Ru')/dx$, i. e.,

$$J(u) = \int_0^1 \left[(P - Q')u^2 - u \frac{d}{dx} (Ru') \right] dx \geq 0.$$

It is now convenient to consider only such variations $u(x)$ that $\int_0^1 u^2(x) dx = 1$. This restriction is exactly analogous to those made in Parts I, II in dealing with the second variation in the algebraic problems, the restrictions $\sum_{i=1}^n v_i^2 = 1$, $\sum_{i=1}^n \eta_i^2 \Delta = 1$, and it should be noted that in no case is this an essential restriction. Then, if ϵ is a positive constant, $J(u) + \epsilon$ is positive:

$$J(u) + \epsilon = \int_0^1 \left[(P - Q' + \epsilon)u^2 - u \frac{d}{dx} (Ru') \right] dx > 0.$$

Introducing the notation

$$L(u) \equiv \frac{d}{dx} (Ru') + (Q' - P - \epsilon)u,$$

we have

$$-\int_0^1 u L(u) dx > 0.$$

Hence, we conclude that the system

$$3) \quad \begin{cases} L(u) = 0, \\ u(0) = u(1) = 0 \end{cases}$$

is incompatible, for if it were compatible, $u(x)$ being a solution, a positive constant c could be found such that $\int_0^1 \bar{u}^2 dx = 1$ and $\int_0^1 \bar{u} L(\bar{u}) dx = 0$, where $\bar{u} = cu$. Let $G(x, \xi)$ be the Green's function of system 3), and let the system

$$\begin{cases} L(u) = \eta(x), \\ u(0) = u(1) = 0, \end{cases}$$

$\eta(x)$ being a continuous function, $0 \leq x \leq 1$, define a variation of the curve $y(x)$; (i. e., a function u such that $\bar{y} - y = \epsilon u$). Then

$$u(x) = \int_0^1 \eta(\xi) G(x, \xi) d\xi$$

and

$$\int_0^1 u L(u) dx = \int_0^1 \int_0^1 \eta(\xi) \eta(x) G(x, \xi) d\xi dx < 0.$$

* See Bolza, "Lectures," p. 52.

† $\psi(u)$ as here used is a standard notation; it should be noted that it is the negative of the $\psi(u)$ used in Part II.

The characteristic numbers of the kernel $G(x, \xi)$ are all negative. For suppose that one, λ_0 , were positive, and that

$$\eta(x) = \lambda_0 \int_0^1 G(x, \xi) \eta(\xi) d\xi.$$

Multiplying both sides by $\eta(x)$ and integrating, we have

$$\int_0^1 \eta^2(x) dx = \lambda_0 \int_0^1 \int_0^1 \eta(\xi) \eta(x) G(x, \xi) d\xi dx.$$

The left-hand side is positive, while the right-hand side is negative, and the contradiction proves the statement. But the characteristic numbers of the kernel $G(x, \xi)$ are the same as for the system*

$$\begin{cases} L(u) = \lambda u, \\ u(0) = u(1) = 0. \end{cases}$$

Then the system

$$4) \quad \begin{cases} L(u) + \lambda u = 0, \\ u(0) = u(1) = 0 \end{cases}$$

has a solution $u(x) \not\equiv 0$ only for positive values of λ , no matter how small a positive constant ε be chosen. Then the system

$$5) \quad \begin{cases} \frac{d}{dx}(Ru') + (Q' - P)u + \lambda u = 0, \\ u(0) = u(1) = 0 \end{cases}$$

has no negative characteristic numbers, for if $\lambda_0 < 0$ were a characteristic number and we choose $\varepsilon < -\lambda_0$, then 4) would have a negative characteristic number, namely, $\lambda_0 + \varepsilon$.

There is one and but one value† of λ for which system 5) has a solution not vanishing in the interval $0 < x < 1$, and we have shown that this value is positive. Then there is no non-identically vanishing solution of the system

$$\begin{cases} -\psi(u) \equiv \frac{d}{dx}(Ru') + (Q' + P)u = 0, \\ u(0) = 0; \quad u(x') = 0, \quad 0 < x' < 1, \end{cases}$$

for if there were, system 5) would have a solution vanishing nowhere in the interval $0 < x < 1$ for λ negative. ‡

* See, for instance, Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," *Göttingische Nachrichten*, 1904, p. 225.

† See Theorem a), Part II.

‡ See Theorem b), Part II.